# Uniqueness conjectures for extended Markov numbers

Matty van Son

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## Introduction

Triples of *regular Markov numbers* are the solutions to the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

These *Markov triples* are the subject of the uniqueness conjecture of regular Markov numbers, introduced by Frobenius in 1913 [7].

**Conjecture 1** (Uniqueness conjecture (Frobenius 1913))**.** Markov triples are uniquely defined by their largest element.

**Example 2.** Both $(1, 5, 2)$ and $(5, 29, 2)$ are Markov triples. By the theory of Markov numbers the number 5 appears in infinitely many Markov triples. The uniqueness conjecture states that the only Markov triple in which 5 is the largest element is $(1, 5, 2)$.

This conjecture is well studied, and shows up in many interesting areas. We refer to the book by Aigner [1] for a general reference. Although the conjecture is not proven, some cases are known, see for example [2, 3].

In this note we extend the uniqueness conjecture for graphs of general Markov numbers, and show that for certain graphs the extended uniqueness conjecture fails (Theorem 13). To define these graphs we first show how regular Markov numbers may be obtained from a graph of sequences.

[15]

## 1 Generalised uniqueness conjecture

Jointly with O. Karpenkov [11] we have developed an extension for regular Markov numbers. We define this extension Subsection 1.1 and introduce the generalised uniqueness conjecture. We develop the first counterexamples to the conjecture in Subsection 1.2.

## 1.1 Development of general Markov numbers

We start with definitions of continued fractions.

**Definition 3.** Let $\alpha = \big(a_i\big)_{i=1}^{n}$ and $\beta = \big(b_i\big)_{i=1}^{m}$ be finite sequences of positive integers. The *concatenation* of $\alpha$ and $\beta$ is $\alpha \oplus \beta = (a_1, \dots, a_n, b_1, \dots, b_m)$. We often shorten the notation $\alpha \oplus \beta$ to $\alpha\beta$.

The *continued fraction expansion of* $\alpha$ is

$$a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}.$$

and is denoted by $[a_1; a_2 : \dots : a_n]$.

Next we define an important notion in the study of Markov numbers and sequences.

**Definition 4.** For a sequence of positive integers $(a_1, \dots, a_n)$ let $c$ and $d$ be the integers with $\gcd(c, d) = 1$ such that

$$[a_1; a_2 : \dots : a_{n-1}] = \frac{c}{d}.$$

Define the *integer sine* of $\alpha$ to be $c$. We use the notation $\breve{K}(\alpha) = c$.

*Remark* 5. The term *integer sine* comes from the study of integer geometry. We recommend the book on this topic by Karpenkov [10] for interested readers.

To calculate the integer sine of a sequence one may evaluate the continued fraction, or a polynomial of elements of the sequence called the *continuant*. For an explanation of continuants see the book by Graham, Knuth, and Patashnik [8].

Aside from integer geometry, Markov numbers have also been studied in relation to hyperbolic geometry. Starting with Cohn [4, 5], works on the topic include Haas [9], Series [12, 13], and more recently Springborn [15].

We define a graph structure that is used to study Markov numbers.

**Definition 6.** Define operations $\mathcal{L}$ and $\mathcal{R}$ on triples of finite sequences of positive integers by

$$\mathcal{L}(\alpha, \gamma, \beta) = (\alpha, \alpha\gamma, \gamma),$$
$$\mathcal{R}(\alpha, \gamma, \beta) = (\gamma, \gamma\beta, \beta).$$

Define a binary graph $\mathcal{G}(\alpha, \beta)$ with root $(\alpha, \alpha\beta, \beta)$, where two vertices $v$ and $w$ are connected by an edge $(v, w)$ if

$$w = \mathcal{L}(v) \quad \text{or} \quad w = \mathcal{R}(v).$$

We define operations $\chi$ and $X$ to obtain a triple graph of positive integers from a graph of triple sequences.

**Definition 7.** Let $\chi$ be the map acting on triples of sequences by

$$\chi(\alpha, \gamma, \beta) = \big(\breve{K}(\alpha), \breve{K}(\gamma), \breve{K}(\beta)\big).$$

Define a map $X$ taking a triple graph of sequences $\mathcal{G}(\alpha, \beta)$ to a triple graph of integers, where vertices $v$ are mapped to $\chi(v)$, and edges $(v, w)$ are mapped to $\big(\chi(v), \chi(w)\big)$. We call the graph

$$\mathcal{T}\big((1,1),(2,2)\big) = X\Big(\mathcal{G}\big((1,1),(2,2)\big)\Big)$$

the *graph of regular Markov numbers.* The triples appearing as vertices in this graph are the solutions to the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

A more complete treatment of the relation between Markov numbers and triple graphs of positive integers, and also triple graphs of $\mathrm{SL}(2, \mathbb{Z})$ matrices and binary quadratic forms, may be found in the papers [11, 16].

Let us collect some known results about the graph of regular Markov numbers.

**Proposition 8.**

*(i) Every triple at a vertex of the graph $\mathcal{T}\big((1,1),(2,2)\big)$ is a Markov triple.*

*(ii) The graph contains every possible Markov triple.*

*(iii) The vertices $v = (a_1, M, a_2)$ with $a_1 \leq M$ and $a_2 \leq M$, and $w = (b_1, Q, b_2)$ with $b_1 \leq Q$ and $b_2 \leq Q$ are connected by an edge $(v, w)$ if and only if either*

$$w = (a_1, 3Ma_1 - a_2, M) \quad \textit{or} \quad w = (M, 3Ma_2 - a_1, a_2).$$

*(iv) The Markov graph is a tree (no loops or double edges).*

This proposition is a collection of classical results in the theory of regular Markov numbers. One may find a proof of each statement in the books by Cusick [6] or Aigner [1].

One may define triple graphs of integers in the same way as Markov numbers but with different sequences. In this note we consider the graphs $\mathcal{T}\big((a,a),(b,b)\big)$ where $a$ and $b$ are positive integers and $a < b$. We call this a *graph of general Markov numbers.* We have the analogue of the uniqueness conjecture for regular Markov numbers.

**Conjecture 9** (Uniqueness conjecture for general Markov numbers)**.** Let $a$ and $b$ be positive integers with $a < b$. Then each triple of integers at a vertex of the graph of general Markov numbers $\mathcal{T}\big((a,a),(b,b)\big)$ is uniquely defined by it's largest element.

| $n$ | $S_n(0)$ | $S_n(1)$ | $b_n/a_n$ |
|---|---|---|---|
| 1 | (4,4) | (11,11) | [2; 1: 3] |
| 2 | (14, 14) | (82, 82) | [5; 1: 6] |
| 3 | (36, 36) | (393, 393) | [10; 1: 11] |
| 4 | (76, 76) | (1364, 1364) | [17; 1: 18] |
| 5 | (140, 140) | (3775, 3775) | [26; 1: 27] |

Table 1: Sequences $S_n(0)$ and $S_n(1)$ for $n = 1, \dots, 5$.

### 1.2 First counterexamples to the general uniqueness conjecture

We define certain graphs of general Markov numbers for which this conjecture is false.

**Definition 10.** For any positive integer $n$ define positive integers $a_n$ and $b_n$ by

$$\begin{aligned} a_n &= n^2 + 3, \\ b_n &= n^4 + 5n^2 + 5. \end{aligned}$$

Note that $\gcd(a_n, b_n) = 1$, and that the ratio $b_n/a_n$ is equal to the continued fraction

$$\frac{b_n}{a_n} = \left[1 + n^2; 1 : 2 + n^2\right]$$

**Definition 11.** Define the sequences $S_n(0)$ and $S_n(1)$ by

$$S_n(0) = (na_n, na_n), \quad S_n(1) = (nb_n, nb_n).$$

We notate the graphs given with these sequences by $\mathcal{T}_n = X\Big(\mathcal{G}\big(S_n(0), S_n(1)\big)\Big)$.

**Example 12.** We show the sequences $S_n(0)$ and $S_n(1)$ for $n = 1, \dots, 5$, along with the continued fraction of $b_n/a_n$, in Table 1.

We present the main result.

**Theorem 13.** *The uniqueness conjecture for general Markov numbers does not hold for any graph $\mathcal{T}_n$, where $n$ is a positive integer.*

In the graph of Markov numbers $\mathcal{T}_n$ for any $n > 0$ there are triples defined

$$\Big(\breve{K}\big(S_n(0)\big), \quad \breve{K}\big(S_n(0)^{5j+1}S_n(1)\big), \quad \breve{K}\big(S_n(0)^{5j}S_n(1)\big)\Big),$$
$$\Big(\breve{K}\big(S_n(0)S_n(1)^{3j}\big), \quad \breve{K}\big(S_n(0)S_n(1)^{3j+1}\big), \quad \breve{K}\big(S_n(1)\big)\Big),$$

for all $j \geq 1$. We show that the largest element of these triples are equal, and hence the uniqueness conjecture for general Markov numbers fails for the graphs $\mathcal{T}_n$. More specifically we have the following proposition.

**Proposition 14.** *For all positive integers $n$ and $j$ we have that*

$$\breve{K}\big(S_n(0)^{5j+1}S_n(1)\big) = \breve{K}\big(S_n(0)S_n(1)^{3j+1}\big).$$

**Example 15.** The simplest examples are in the graph $\mathcal{T}_1$, which contains the triples

$$\Big(\breve{K}(4,4),\ \breve{K}\big((4,4)^6 \oplus (11,11)\big),\ \breve{K}\big((4,4)^5 \oplus (11,11)\big)\Big),$$
$$\Big(\breve{K}\big((4,4) \oplus (11,11)^3\big),\ \breve{K}\big((4,4) \oplus (11,11)^4\big),\ \breve{K}(11,11)\Big),$$

which, when evaluated, give

$$(4, 355318099,\ 19801199),$$
$$(2888956, 355318099,\ 11).$$

## 1.3 Proof of Theorem 13

Theorem 13 follows from Proposition 14. To prove this proposition we first define sequences of positive integers $\big(L_n(j)\big)_{j>0}$ and $\big(R_n(j)\big)_{j>0}$ containing the values

$$\breve{K}\big(S_n(0)^{5j+1}S_n(1)\big) \quad \text{and} \quad \breve{K}\big(S_n(0)S_n(1)^{3j+1}\big).$$

We show in Lemmas 21 and 22 that both $\big(L_n(j)\big)_{j>0}$ and $\big(R_n(j)\big)_{j>0}$ are subsequences of another sequence $\big(A_n(j)\big)_{j>0}$ for every $n > 0$. Then we show that their elements align within $\big(A_n(j)\big)_{j>0}$ in such a way that Proposition 14 holds.

**Definition 16.** Let $n$ be a positive integer and let $a_n = n^2+3$ and $b_n = n^4+5n^2+5$. Define

$$\begin{aligned} l_n &= (na_n)^2+2, \quad r_n = (nb_n)^2+2, \\ L_n(1) &= \breve{K}(na_n, na_n, nb_n, nb_n), \quad L_n(2) = \breve{K}(na_n, na_n, na_n, na_n, nb_n, nb_n), \\ R_n(1) &= \breve{K}(na_n, na_n, nb_n, nb_n), \quad R_n(2) = \breve{K}(na_n, na_n, nb_n, nb_n, nb_n, nb_n). \end{aligned}$$

For $j > 2$ define

$$L_n(j) = l_nL_n(j-1) - L_n(j-2) \quad \text{and} \quad R_n(j) = r_nR_n(j-1) - R_n(j-2).$$

We relate the sequences $\big(L_n(j)\big)_{j>0}$ and $\big(R_n(j)\big)_{j>0}$ to the numbers in Theorem 13.

**Proposition 17.** *The following statements are equivalent:*

(*i*) *For all positive integers $n$ and $i$ we have that*

$$\breve{K}\big(S_n(0)^{5i+1} \oplus S_n(1)\big) = \breve{K}\big(S_n(0) \oplus S_n(1)^{3i+1}\big).$$

(*ii*) *For all positive integers $n$ and $i$ we have that*

$$L_n(5i+1) = R_n(3i+1).$$

The proof of this proposition relies on the recurrence relation for general Markov numbers which may be found in [11, Theorem 7.15].

**Example 18.** The first 6 elements of the sequences $\big(L_1(j)\big)_{j>0}$ and $\big(R_1(j)\big)_{j>0}$ are

$$\begin{aligned}\big(L_1(j)\big)_{j=1}^{6} &= (191, 3427, 61495, 1103483, 19801199, 355318099),\\ \big(R_1(j)\big)_{j=1}^{6} &= (191, 23489, 2888956, 355318099, 43701237221, 5374896860084).\end{aligned}$$

Next we define the sequence $\big(A_n(j)\big)_{j>0}$.

**Definition 19.** Let $A_n(1) = 1$ and $A_n(2) = n(n^2+4)$. Then for $j > 2$ define

$$A_n(j) = nA_n(j-1) + A_n(j-2).$$

*Remark* 20. We guessed the structure of this sequence from looking at the case for $n = 1$, where $\big(A_1(j)\big)_{j>0}$ is the sequence A022095 in [14].

**Lemma 21.** *For all positive integers $n$ and $j$ we have that*

$$R_n(j) = A_n(10j).$$

*Proof.* For any $n > 0$ we have $na_n = 3n+n^3$, $b = n^5+5n^3+5n$, and also that $r_n = (nb_n)^2+2$. Hence

$$R_n(1) = \breve{K}(na_n, na_n, nb_n, nb_n) = n^{11} + 11n^9 + 44n^7 + 76n^5 + 51n^3 + 8n.$$

By computation we see that $A_n(10) = R_n(1)$. Also we see that

$$\begin{aligned}R_n(2) =& n^{21} + 21\,n^{19} + 189\,n^{17} + 951\,n^{15} + 2926\,n^{13} + 5655\,n^{11} +\\ & 6787\,n^9 + 4818\,n^7 + 1827\,n^5 + 301\,n^3 + 13\,n = A_n(20).\end{aligned}$$

This serves as a base of induction.

Assume that $A_n(10j) = R_n(j)$ for all $j = 1, \ldots, k-1$, for some $k > 2$. Then

$$\begin{aligned}R_n(k) &= r_nR_n(k-1) - R_n(k-2)\\ &= r_nA_n\big(10(k-1)\big) - A_n\big(10(k-2)\big),\\ &= r_nA_n(10k-10) - A_n(10k-20),\end{aligned}$$

with the first equality following definition, and the second equality following the induction hypothesis. Note that

$$\begin{aligned}
A_n(10k-10) &= nA_n(10k-10-1) + A_n(10k-10-2) \\
&= (n^2+1)A_n(10k-10-2) + A_n(10k-10-3) \\
&\vdots \\
&= x_1 A_n(10k-10-9) + x_2 A_n(10k-10-10),
\end{aligned}$$

where $x_1$ and $x_2$ are given through direct computation (we used Maple software) by

$$\begin{aligned}
x_1 &= n^9 + 8\,n^7 + 21\,n^5 + 20\,n^3 + 5\,n, \\
x_2 &= n^8 + 7\,n^6 + 15\,n^4 + 10\,n^2 + 1.
\end{aligned}$$

In the same manner we have

$$\begin{aligned}
A_n(10k) &= nA_n(10k-1) + A_n(10k-2) \\
&= (n^2+1)A_n(10k-2) + A_n(10k-3) \\
&\vdots \\
&= y_1 A_n(10k-19) + y_2 A_n(10k-20),
\end{aligned}$$

where $y_1$ and $y_2$ are given through direct computation by

$$\begin{aligned}
y_1 =& n^{19} + 18\,n^{17} + 136\,n^{15} + 560\,n^{13} + 1365\,n^{11} + \\
& 2002\,n^9 + 1716\,n^7 + 792\,n^5 + 165\,n^3 + 10\,n, \\
y_2 =& n^{18} + 17\,n^{16} + 120\,n^{14} + 455\,n^{12} + 1001\,n^{10} + \\
& 1287\,n^8 + 924\,n^6 + 330\,n^4 + 45\,n^2 + 1.
\end{aligned}$$

Using this information we have that

$$\begin{aligned}
R_n(k) &= r_n A_n(10k-10) - A_n(10k-20) \\
&= r_n\big(x_1 A_n(10k-10-9) + x_2 A_n(10k-10-10)\big) - A_n(10k-20) \\
&= r_n x_1 A_n(10k-19) + (r_n x_2 - 1)A_n(10k-20).
\end{aligned}$$

Through direct computation we see that $y_1 = r_n x_1$ and $y_2 = r_n x_2 - 1$. Hence we have that

$$R_n(k) = A_n(10k),$$

so the induction holds and the proof is complete. $\square$

Lemma 21 says that $\big(R_n(j)\big)_{j>0}$ is a subsequence of $\big(A_n(j)\big)_{j>0}$. Now we show an analogous statement for $\big(L_n(j)\big)_{j>0}$.

**Lemma 22.** *For all positive integers $n$ and $i$ we have that*

$$L_n(j) = A_n\big(10 + 6(j-1)\big).$$

*Proof.* We use induction. We have that $A_n(10) = L_n(1)$, as in the proof for Lemma 21 . We also have $l_n = (n^3+3n)^2{+}2$, and that

$$\begin{aligned} A_n(16) &= n^{17} + 17n^{15} + 119n^{13} + 441n^{11} + 925n^9 + 1086n^7 + 658n^5 + 169n^3 + 11n \\ &= L_n(2). \end{aligned}$$

This is our base of induction. Assume that $A_n\big(10+6(j-1)\big) = A_n(6j+4) = L_n(j)$ for all $j = 1, \ldots, k{-}1$, for some $k > 2$. Then

$$\begin{aligned} L_n(k) &= l_n L_n(k-1) - L_n(k-2) \\ &= l_n A_n(6k-2) - A_n(6k-8), \end{aligned}$$

with the first equality following definition, and the second equality following the induction hypothesis. In a similar way to the proof for $\big(R_n(j)\big)_{j>0} = \big(A_n(j)\big)_{j>0}$ we have that

$$A_n(6k-2) = w_1 A_n(6k-7) + w_2 A_n(6k-8),$$

where $w_1 = n^5 + 4n^3 + 3n$ and $w_2 = n^4 + 3n^2 + 1$. Also we have

$$A_n\big(10+6(k-1)\big) = A_n(6k+4) = z_1 A_n(6k-7) + z_2 A_n(6k-8),$$

where

$$\begin{aligned} z_1 &= n^{11} + 10\,n^9 + 36\,n^7 + 56\,n^5 + 35\,n^3 + 6\,n, \\ z_2 &= n^{10} + 9\,n^8 + 28\,n^6 + 35\,n^4 + 15\,n^2 + 1. \end{aligned}$$

Now we have that

$$\begin{aligned} L_n(k) &= l_n A_n(6k-2) - A_n(6k-8), \\ &= l_n w_1 A_n(6k-7) + (l_n w_2 - 1) A_n(6k-8), \end{aligned}$$

and by direct computation we see that $z_1 = l_n w_1$ and $z_2 = l_n w_2 - 1$.

So $L_n(k) = A_n\big(10+6(k-1)\big)$ and induction holds. This completes the proof. □

We prove Proposition 14.

*Proof of Proposition 14.* Given Lemmas 21 and 22 we need only show the values $L_n(5j+1)$ and $R_n(3j+1)$ align within the sequence $\big(A_n(j)\big)_{j>0}$. Indeed we have that

$$\begin{aligned} L_n(5j+1) &= A_n\big(10+6(5j+1-1)\big) = A_n(30j+10), \\ R_n(3j+1) &= A_n\big(10(3j+1)\big) = A_n(30j+10). \end{aligned}$$

Hence the claim is proved. □

Theorem 13 follows as a corollary.

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